ANGULAR MOMENTUM AND THE ALGEBRA OF CURRENT COMPONENTS*

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It has recently been proposed[1] that the space integrals of all components of the vector current octet $\mathfrak{F}_{i\alpha}$ and the axial vector current octet $\mathfrak{F}_{i\alpha}{}^5$ obey the same commutation rules as if the currents were equal to $\mathcal{G}_{i\alpha}$ and $\mathcal{G}_{i\alpha}{}^5$, respectively, where, in terms of a quark field q, we have

$$\mathcal{G}_{i\alpha} \equiv i\bar{q}\gamma_\alpha\lambda_i q/2, \quad \mathcal{G}_{i\alpha}{}^5 \equiv i\bar{q}\gamma_\alpha\gamma_5\lambda_i q/2. \tag{1}$$

The system is algebraically closed if a ninth pair of currents is added; thus we let the index i run from 0 to 8, with $\lambda_0 = (\frac{2}{3})^{1/2}1$. We may rearrange the 72 components of $\int \mathcal{G}_{i\alpha}d^3x$ and $\int \mathcal{G}_{i\alpha}{}^5 d^3x$ to obtain the 72 Hermitian operators

$$G_{ij}{}^{\pm} = (\tfrac{3}{32})^{1/2}\int q^\dagger \lambda_i \sigma_j (1\pm\gamma_5) q d^3x \tag{2}$$
$$(i=0,\cdots,8;\ j=0,\cdots,3),$$

where $\sigma_0 \equiv 1$. The corresponding linear combinations of the $\int \mathfrak{F}_{i\alpha}d^3x$ and $\int \mathfrak{F}_{i\alpha}{}^5 d^3x$ are called $A_{ij}{}^{\pm}$; either the $G_{ij}{}^{\pm}$ or the $A_{ij}{}^{\pm}$ generate the algebra of $U(6)\otimes U(6)$.[2]

It was suggested that the $A_{ij}{}^{+}$ and $A_{ij}{}^{-}$ form a system of very approximate symmetries of the hadrons. In the energy density $\mathcal{H} = -\theta_{44}$, there is a large term $\mathcal{H}'$ that breaks the symmetry down to that of the quantities

$$A_{ij} \equiv A_{ij}{}^{+} + A_{ij}{}^{-} \tag{3}$$
$$(i=0,\cdots,8;\ j=0,\cdots,3),$$

which generate the algebra of U(6). The A_{ij} are apparently a good set of hadron symmetries, and we have thus interpreted the algebra of U(6) discovered by Gürsey and Radicati,[3] Sakita,[4] and Zweig[5] and further developed in several recent papers.[6-12] In this Letter we consider the relation of the algebra to the angular momentum $\vec{J}$, as well as the manner in which the symmetry U(6) is broken down to U(3). [The way in which U(3) is violated will be treated elsewhere.] Some features of the U(6) theory that have been obscure are clarified here.

Among the generators A_{ij}, we note that A_{00} is proportional to the baryon number (or the triplet number if the triplet q does not refer to quarks) and that it commutes with the other 35 generators, which give SU(6). The generators A_{i0} ($i=1,\cdots,8$) are just $(\frac{3}{2})^{1/2}F_i$, where the F_i are the components of the F spin connected with SU(3) symmetry. The generators[13]

$$S_j \equiv A_{0j} \quad (j=1,2,3)$$

act exactly like a spin angular momentum since they obey the rules

$$[S_i, S_j] = ie_{ijk}S_k; \quad [J_i, S_j] = ie_{ijk}S_k. \tag{4}$$

(In the quark model, of course, $\vec{S}$ is just the total spin of quarks, including any number of quark-antiquark pairs.) Naturally, $\vec{S}$ is not equal to the total angular momentum $\vec{J}$, since it does not include the orbital angular momentum; we may define the difference

$$\vec{L} = \vec{J} - \vec{S}, \tag{5}$$

and note that $\vec{L}$ obeys the rules

$$[L_i, L_j] = ie_{ijk}L_k, \ [J_i, L_j] = ie_{ijk}L_k, \ [L_i, S_j] = 0, \tag{6}$$

by virtue of (4). Thus, $\vec{L}$ acts like an orbital angular momentum.[14] (In a pure quark model, that is what it is. In a model with quarks and other basic particles, $\vec{L}$ would include the intrinsic spins of the extra particles.)

It should perhaps be emphasized that the remaining generators A_{ij} ($i=1,\cdots,8$; $j=1,\cdots,3$) are not proportional to F_iS_j, but they do transform like F_iS_j under the group SU(6).

In the approximation of conservation of all the A_{ij}, each degenerate particle multiplet at rest belongs to a definite SU(6) representation and a definite value of L, with $\vec{L}$ and $\vec{S}$ added vectorially to give the spins J of the particles.[15] The vector meson octet and singlet and the pseudoscalar meson octet presumably belong to $\underline{35}$ with $L=0$; the pseudoscalar meson singlet may belong to $\underline{1}$ with $L=0$; a $J=2^+$ meson singlet might belong to $\underline{1}$ with $L=2$; and so forth. The baryon $J=\frac{1}{2}^+$ octet and $J=\frac{3}{2}^+$ decimet presumably belong to $\underline{56}$ with $L=0$.

We may now discuss Regge recurrences. The first recurrence of the $\underline{56}$, for example, has $L=2$ added vectorially to $S=\frac{1}{2}$ for the octet and $S=\frac{3}{2}$ for the decimet; altogether, this multiplet has $5\times 56=280$ states. The Regge recurrences of $J=\frac{1}{2}^+$ ($\underline{8}$) and $J=\frac{3}{2}^+$ ($\underline{10}$), with $J=\frac{5}{2}^+$ ($\underline{8}$) and $J=\frac{7}{2}^+$ ($\underline{10}$), respectively, are included; but so are particles with $J=\frac{3}{2}^+$ ($\underline{8}$) and with $J=\frac{5}{2}^+$ ($\underline{10}$), $\frac{3}{2}^+$ ($\underline{10}$), and $\frac{1}{2}^+$ ($\underline{10}$), all of which lie on trajectories that give "nonsense" (no real particles) at $L=0$.

In a quark model, there is a term in the energy density $\mathcal{H}=-\theta_{44}$, namely,

$$(2i)^{-1}(q^\dagger\vec{\alpha}\cdot\nabla q-\nabla q^\dagger\cdot\vec{\alpha}q), \tag{7}$$

that breaks SU(6) down to SU(3). Let us assume that in the true theory the term $\mathcal{H}''$ that breaks SU(6) down to SU(3) transforms in the same manner, namely ($\underline{1}$, $\underline{35}$) and ($\underline{35}$, $\underline{1}$) under SU(6) $\otimes$ SU(6), or $\underline{35}$ under SU(6), with $L=1$. In first order, such a term $\mathcal{H}''$ can lead to an $\vec{L}\cdot\vec{S}$ splitting in a multiplet with $L>0$, such as the Regge recurrence of the $\underline{56}$ discussed above. In second order, $\mathcal{H}''$ can split a multiplet with $L=0$; we obtain splittings that transform like $\underline{405}$ and $\underline{189}$ under SU(6). These are the only representations in $\underline{35}\otimes\underline{35}$, besides the trivial representation $\underline{1}$, that contain a unitary singlet with $S=0$.

Bég and Singh[9] show that unitary singlet mass perturbations belonging to $\underline{405}$ and $\underline{189}$ are sufficient to explain the splitting between octet and decimet in $\underline{56}$ and between pseudoscalar and vector mesons in $\underline{35}$. In order to explain the φ-ω degeneracy before SU(3) breaking, we must have a particular linear combination of $\underline{405}$ and $\underline{189}$ that gives spin splitting but no unitary-spin splitting; such a combination cannot be required by symmetry under U(6) alone, but must be explained by approximate symmetry under a larger algebra, such as that of U(6) $\otimes$ U(6).

An interesting application of the hypothesis that $\mathcal{H}''$ transforms like $\underline{35}$ with $L=1$ is the study of the magnetic moment operator μ_j between states with $L=0$. The operator μ_j transforms like $e_{jkl}\int x_k \mathfrak{F}_{el} d^3x$, where the index e refers to the charge direction in SU(3) space. Evidently, μ_j belongs to $\underline{35}$ with $L=1$ and in the limit of U(6) symmetry it gives zero between $L=0$ states. To first order in $\mathcal{H}''$, we can get an effective μ_j operator that has $L=0$ and transforms under U(6) like $\underline{35}\otimes\underline{35}$; it contains pieces that belong to $\underline{35}$, $\underline{189}$, $\underline{405}$, $\underline{280}$, and $\underline{280}^*$. Now between the baryon multiplet $\underline{56}$ ($L=0$) and itself, the only pieces that can contribute are $\underline{35}$ and $\underline{405}$, which are contained in $\underline{56}\otimes\underline{56}^*$; the effective μ_j in this case has the form

$$\mu_j=a_1A_{ej}+a_2e_{jkl}\{A_{ek},A_{0l}\}. \tag{8}$$

The second term can be shown to vanish by time-reversal invariance. Thus the nucleon magnetic moments obey the rule $\mu_n/\mu_p=-\frac{2}{3}$ characteristic of the first term alone. This ratio was first presented in reference 11, where it was not explained why the effective μ_j transforms in this case according to $\underline{35}$ alone.

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[11]M. A. B. Bég, B. W. Lee, and A. Pais, Phys. Rev. Letters $\underline{13}$, 514 (1964).

[12]H. J. Lipkin, Phys. Rev. Letters $\underline{13}$, 590 (1964).

[13]The symbol S is used for something different in references 9 and 10.

[14]We note that $[J_l, A_{ij}{}^{\pm}]=[S_l, A_{ij}{}^{\pm}]=ie_{ljk}A_{ik}{}^{\pm}$, so that $\vec{L}$ commutes with all components of $A_{ij}{}^+$ and $A_{ij}{}^-$.

[15]For each value of L_3, there must be a complete representation of the algebra, including all values of S_3; thus we obtain all the values of J that come from vector addition. If we consider a different Lorentz frame, the definition of the A_{ij} (including the components of $\vec{S}$) changes accordingly and so do the definitions of $\vec{L}$ and $\vec{J}$. In the new frame, we again consider states of hadrons at rest and apply the new operators to them.